On The New Exact Solutions for the Nonlinear Models Arising In Plasma Physics

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ABSTRACT

In this article we apply the Riccati-Bernoulli Sub-ODE method in order to find the exact traveling wave solutions for the Bogoyavlenskii equation and the (3 + 1)-dimensional mKdV-ZK equation. The travelling wave solutions of many equations physically and mathematically are expressed by rational functions, trigonometric functions and hyperbolic functions. Three-dimensional graphics of some solutions have been plotted. All the computations and graphic plots are performed with the aid of the Matlab package. We finally compare our results with the other methods and clarify that the Riccati-Bernoulli Sub-ODE method superior to other methods.

Keywords: Riccati-Bernoulli Sub-ODE Method; The Bogoyavlenskii Equation; The (3 + 1)Dimensional Mkdv-ZK Equation; B¨Acklund Transformation; Traveling Wave Solutions

INTRODUCTION

Nonlinear phenomena arise in many areas of science, for example fluid mechanics, biology, optical fibers, plasma physics, chemical kinematics and so on. In fact these phenomena are reflected in interesting Nonlinear Partial Differential Equations (NPDEs), see [1-12]. Due to the importance of these equations, many researchers are concentrated on soliton solutions of them. Therefore investigating new methods to solve more complicated problems. Thus, many new methods have been proposed, such as the tanh-sech method [13-15], Jacobi elliptic function method [16-18], exp-function method [19, 20], sine-cosine method [21-23], homogeneous balance method [24, 25], F-expansion method [26-28], extended tanh-method [29, 30], $(\frac{G'}{G})-$ expansion method [31, 32] and so on.

The present article is devoted to construct the exact solutions for the Bogoyavlenskii equation and the (3 + 1)-dimensional mKdV-ZK equation, using the Riccati-Bernoulli sub-ODE method [11,12,33]. By using a traveling wave transformation and the Riccati-Bernoulli equation, these equations can be transferred to a set of algebraic equations. Hence we get the exact solutions of the previous NPDEs, by solving these algebraic equations. If we get a solution of these NPDEs, we obtain new infinite sequence of solutions of these equations by using a B¨acklund transformation. Actually, we give new solutions and show that this method is efficient, robust and adequate for solving other type of NPDEs. Moreover the Riccati-Bernoulli Sub-ODE method has an interesting feature, namely it is give infinite sequence of solutions. Indeed all presented solutions have so important contribution for the explanation of some practical physical problems.

The novelties of this paper are mainly exhibited in three aspects: first, we apply a new method, which is not so familiar, a namely the Riccati-Bernoulli Sub-ODE method [11,12, 33]. The Bogoyavlenskii equation and the (3 + 1)-dimensional mKdV-ZK equation are chosen to illustrate the validity of this method. Second, we also show that the proposed method gives infinite sequence of solutions, using a B¨acklund transformation. Third, we obtain new types of exact analytical solutions. Moreover comparing our results with other results, one can see that our results are new and most extensive.

The remaining part of this paper is organized as follows: In section 2 we recall the RiccatiBernoulli Sub-ODE method and give a B¨acklund transformation of the Riccati-Bernoulli equation. In Section 3, we apply the Riccati-Bernoulli sub-ODE method to solve the Bogoyavlenskii equation and the (3 + 1)-dimensional mKdV-ZK equation. In Section 4, some two and threedimensional graphs of some solutions are given. In section 5 we present some interesting applications of the solutions for the Bogoyavlenskii equation and the (3 + 1)-dimensional mKdV-ZK equation. In section 6, there is a comparison and investigation between our results and the other methods. Conclusion is in 7.

DESCRIPTION OF METHOD

Step 1. Consider the following nonlinear evolution equation

$$H(\vartheta,\vartheta_t,\vartheta_x,\vartheta_{tt},\vartheta_{xx},....) = 0, \quad (2.1)$$

where H is a polynomial in $\vartheta(x,t)$ and its partial derivatives with even highest order derivatives and nonlinear terms. The main steps of this method [33] given as follows: Step 1. We use the wave transformation

$$\vartheta(x,t) = \vartheta(\xi), \qquad \xi = k(x + vt), \qquad (2.2)$$

where the localized wave solution $\vartheta(\xi)$ travels with speed v and k is a positive constant. Using equation (2.2), one can transform equation (2.1) into the following ODE:

$$D(\vartheta,\vartheta',\vartheta'',\vartheta''',.....) = 0, \qquad (2.3)$$

where D is a polynomial in $\vartheta(\xi)$ and its total derivatives, while $\vartheta'(\xi) = \frac{d\vartheta}{d\xi},\ \vartheta''(\xi) = \frac{d^2\vartheta}{d\xi^2}$ and so on.

Step 2. We assume that equation (2.3) has the formal solution in the following form:

$$\vartheta' = a\vartheta^{2-n} + b\vartheta + c\vartheta^n, \qquad (2.4)$$

where a,b,c and n are constants to be determined in sequel .

From equation (2.4), we get $\vartheta''= ab(3 - n)\vartheta^{2-n} + a^2(2 - n)\vartheta^{3-2n} + nc^2\vartheta^{2n-1} + bc(n + 1)\vartheta^n + (2ac + b^2)\vartheta$, (2.5)

$$\vartheta''' = (ab(3 - n)(2 - n)\vartheta^{1-n} + a^2(2 - n)(3 - 2n)\vartheta^{2-2n} + n(2n - 1)c^2\vartheta^{2n-2} + bcn(n + 1)\vartheta^{n-1} + (2ac + b^2))\vartheta'. \qquad (2.6)$$

CLASSIFICATION OF THE SOLUTIONS

Here we give the cases of solutions for the Riccati-Bernoulli equation (2.4).

Case 1: When n = 1, the solution of equation (2.4) is

$$\vartheta(\xi) = \mu e^{(a+b+c)\xi}.$$

Case 2: For n ≠1, b = 0 and c = 0, the solution is (2.7)

$$\vartheta(\xi) = (a(n-1)(\xi+\mu))^{\frac{1}{n-1}} \qquad (2.8)$$

Case 3: For n ≠ 1, b ≠ 0 and c = 0, the solution is

$$\vartheta(\xi) = \left(\frac{-a}{b} + \mu e^{b(n-1)\xi}\right)^{\frac{r}{n-1}} \qquad (2.9)$$

Case 4: For $n \neq 1, a \neq 0$ and $b^2 - 4ac < 0$, the solutions are

$$\vartheta(\xi) = \left(\frac{-b}{2a} + \frac{\sqrt{4ac-b^2}}{2a} tan\left(\frac{(1-n)\sqrt{4ac-b^2}}{2}(\xi+\mu)\right)\right)^{\frac{1}{1-n}} \quad (2.10)$$

and

$$\vartheta(\xi) = \left(\frac{-b}{2a} - \frac{\sqrt{4ac-b^2}}{2a} cot\left(\frac{(1-n)\sqrt{4ac-b^2}}{2}(\xi+\mu)\right)\right)^{\frac{1}{1-n}} \quad (2.11)$$

Case 5: For $n \neq 1, a \neq 0$ and $b^2 - 4ac > 0$, the solutions are

$$\vartheta(\xi) = \left(\frac{-b}{2a} - \frac{\sqrt{b^2-4ac}}{2a} coth\left(\frac{(1-n)\sqrt{b^2-4ac}}{2}(\xi+\mu)\right)\right)^{\frac{1}{1-n}} \quad (2.12)$$

and

$$\vartheta(\xi) = \left(\frac{-b}{2a} - \frac{\sqrt{b^2-4ac}}{2a} tanh\left(\frac{(1-n)\sqrt{b^2-4ac}}{2}(\xi+\mu)\right)\right)^{\frac{1}{1-n}} \quad (2.13)$$

Case 6: For $n \neq 1, a \neq 0$ and $b^2 - 4ac = 0$, the solution is

$$\vartheta(\xi) = \left(\frac{1}{a(n-1)(\xi+\mu)} - \frac{b}{2a}\right)^{\frac{1}{1-n}} \quad (2.14)$$

where μ is an arbitrary constant.

Step 3. Superseding the derivatives of ϑ into equation (2.3) gives an algebraic equation of ϑ. Noticing the symmetry of the right-hand item of equation (2.4) and setting the highest power exponents of ϑ to equivalence in equation (2.3), n can be determined. Comparing the coefficients of ϑi yields a set of algebraic equations for a,b,c,and v. Solving these algebraic equations and substituting n,a,b,c,v, and ξ = k(x + vt) into equations (2.7)-(2.14)), we have traveling wave solutions of equation (2.1).

BACKLUND TRANSFORMATION

When ϑm−1(ξ) and ϑm(ξ)(ϑm(ξ) = ϑm(ϑm−1(ξ))) are the solutions of equation (2.4), we get

$$\frac{d\vartheta_m(\xi)}{d\xi} = \frac{d\vartheta_m(\xi)}{d\vartheta_{m-1}(\xi)}\frac{d\vartheta_{m-1}(\xi)}{d\xi} = \frac{d\vartheta_m(\xi)}{d\vartheta_{m-1}(\xi)}(a\vartheta_{m-1}^{2-n} + b\vartheta_{m-1} + c\vartheta_{m}^{n}$$

, namely

$$\frac{d\vartheta_m(\xi)}{a\vartheta_m^{2-n} + b\vartheta_m + c\vartheta_m^n} = \frac{d\vartheta_m(\xi)}{a\vartheta_{m-1}^{2-n} + b\vartheta_{m-1} + c\vartheta_{m-1}^n} \quad (2.15)$$

Integrating equation (2.15) once with respect to ξ, we get

$$\vartheta_m(\xi) = \left(\frac{-cK_1 + aK_2(\vartheta_{m-1}(\xi))^{1-n}}{bK_1 + aK_2 + aK_1(\vartheta_{m-1}(\xi))^{1-n}}\right)^{\frac{1}{1-n}} \quad (2.16)$$

where K1 and K2 are arbitrary constants.

Equation (2.16) is a B¨acklund transformation of equation (2.4). If we obtain a solution of this equation, we use equation (2.16) in order to get infinite sequence of solutions of equation (2.4), and like wise of equation (2.1).

APPLICATIONS

The Bogoyavlenskii equation

Here we solve the Bogoyavlenskii equation, see [34, 35, 36], which given as follows:

$$4u_t + u_{xxy} - 4u^2u_y - 4u_xv = 0, \quad (3.1) \quad uu_y = v_x,$$

This equation describe the (2+1)-dimensional interaction of a Riemann wave propagating along the y axis with a long wave along the x axis. The differentiable functions u = u(x,y,t) and v = v(x,y,t) in the independent variables x, y and t, represent the physical field and some potential, respectively.

We seek the traveling wave solution for equation (3.1) in the form

$$u(x,y,t) = u(\xi), \quad v(x,y,t) = v(\xi), \quad \xi = x + y - wt. \quad (3.2)$$

where w is real constant.

Substituting equation (3.2) into equation (3.1) and integrating once the second equation of equation (3.1) and for simplicity, equating the integration constant equal to zero, we have

$$-4wu' + u^m - 4u^2u' - 4u'v = 0 = 0, \quad (3.3)$$

$$\frac{u^2}{2} = v.$$

Substituting the second equation of (3.3) into the first equation, after integrating once the resultant, yields

$$u'' - 4wu - 2u^3 = 0. \quad (3.4)$$

Substituting equations (2.5) into equation (3.4), we obtain

$$(ab(3-n)u^{2-n} + a^2(2-n)u^{3-2n} + nc^2u^{2n-1} + bc(n+1)u^n + (2ac+b^2)u) - 4wu - 2u^3 = 0. \quad (3.5)$$

Setting n = 0, equation (3.5) is reduced to

$$3ab\,u^2 + 2a^2u^3 + bc + (2ac+b^2)u) - 4wu - 2u^3 = 0 \quad (3.6)$$

Setting each coefficient of ui(i = 0,1,2,3) to zero, we get (3.7)

$$bc = 0,$$

$$2ac + b^2) - 4w = 0$$

$$ab = 0,$$

$$a^2 - 1 = 0. \quad (3.8)$$

Solving equations (3.7)-(3.10), we get

b = 0,(3.11) c = ±2w, (3.12) a = ±1. (3.13)

Rational function solutions: (When b = 0 and c = 0, i.e., w = 0)

Substituting equations (3.11)-(3.13) and (3.2) into equations (2.8), we obtain the exact wave solutions of equation (3.1),

$$u_1(x,t) = (-a(x+y-wt+\mu))^{-1}, \quad (3.14)$$

$$v_1(x,t) = \frac{1}{2}\left((-a(x+y-wt+\mu))^{-1}\right)^2 \quad (3.15)$$

Trigonometric function solution: (When w < 0)

Substituting equations (3.11)-(3.13) and (3.2) into equations (2.10) and (2.11), we obtain the exact wave solutions of equation (3.1),(3.16)

$$u_{2,3}(x,y,t) = \pm\sqrt{-2w}\; tan\left(\sqrt{-2w}(x+y-wt+\mu)\right)$$

$$v_{2,3}(x,y,t) = -w\; tan^2\left(\sqrt{-2w}(x+y-wt+\mu)\right), \quad (3.17)$$

and (3.18)

$$u_{4,5}(x,y,t) = \pm\sqrt{-2w}\; cot\left(\sqrt{-2w}(x+y-wt+\mu)\right) \quad (3.19)$$

$$v_{4,5}(x,y,t) = -w\; cot^2\left(\sqrt{-2w}(x+y-wt+\mu)\right),$$

where w and μ are arbitrary constants.

Hyperbolic function solution : (When w > 0)

Substituting equations (3.11)-(3.13) and (3.2) into equations (2.12) and (2.13), we obtain the exact wave solutions of equation (3.1). (3.20)

$$u_{6,7}(x,y,t) = \pm\sqrt{2w}\; tanh\left(\sqrt{2w}(x+y-wt+\mu)\right)$$

$$v_{6,7}(x,y,t) = w\; tanh^2\left(\sqrt{2w}(x+y-wt+\mu)\right)$$

$$u_{8,9}(x,y,t) = \pm\sqrt{2w}\; coth\left(\sqrt{2w}(x+y-wt+\mu)\right)$$

$$v_{8,9}(x,y,t) = w\; coth^2\left(\sqrt{2w}(x+y-wt+\mu)\right).$$

(3.21),(3.22) and (3.33)

where w and μ are arbitrary constants.

Remark 3.1. Applying equation (2.16) to ui(x,y,t), i =1,2,...,9, we obtain an infinite sequence of solutions of equation (3.4). Consequently, we get an infinite sequence of solutions of equation (3.1). For illustration, by applying equation (2.16) to ui(x,y,t), i =1,2,...,9, once, we have new solutions of equation (3.4)

$$u_1^\star(x,y,t) = \frac{C_3}{-aC_3(x+y-wt+\mu) \pm 1},$$

$$u_{2,3}^\star(x,y,t) = \frac{\pm 2w \pm C_3\sqrt{-2w}\; tan\left(\sqrt{-2w}(x+y-wt+\mu)\right)}{C_3 \pm \sqrt{-2w}\; tan\left(\sqrt{-2w}(x+y-wt+\mu)\right)}$$

$$u_{4,5}^\star(x,y,t) = \frac{\pm 2w \pm C_3\sqrt{-2w}\; cot\left(\sqrt{-2w}(x+y-wt+\mu)\right)}{C_3 \pm \sqrt{-2w}\; cot\left(\sqrt{-2w}(x+y-wt+\mu)\right)},$$

$$u^*_{6,7}(x,y,t) = \pm 2w \pm C_3\sqrt{2w}\, tanh\sqrt{2w}(x+y-wt+\mu) / C_3 \pm \sqrt{-2w}\, tanh\sqrt{-2w}(x+y-wt+\mu), \quad (3.27)$$

$$u^*_{8,9}(x,y,t) = \pm 2w \pm C_3\sqrt{2w}\, coth\sqrt{2w}(x+y-wt+\mu) / C_3 \pm \sqrt{-2w}\, coth\sqrt{-2w}(x+y-wt+\mu), \quad (3.28)$$

where C_3,w,μ are arbitrary constants.

The (3 + 1)-dimensional mKdV-ZK equation

Here we solve the (3 + 1)-dimensional mKdV-ZK equation, [38-40]. This equation given as follows:

$$\chi_t + \alpha(\chi^2)\chi_x + \chi_{xxx} + \chi_{xyy} + \chi_{xzz} = 0, \quad (3.29)$$

where α is a nonzero constant, t is the scaled time coordinate, x, y and z are the scaled space coordinates. This nonlinear equation describes the nonlinear behaviors of ion-acoustic waves in a magnetized plasma where the cooler ions are treated as a fluid with adiabatic pressure and the hot isothermal electrons are described by a Boltzman distribution [41].

Using the traveling wave transformation

$$\chi(x,y,z,t) = \chi(\zeta), \quad \zeta = x + y + \epsilon z - \lambda t, \quad (3.30)$$

where and λ are real constants.

Equation (3.29) transforms into the following ODEs, using (3.30):

$$3(\epsilon^2 + 2)\chi^n - 3\lambda\chi + \alpha\chi^3 = 0. \quad (3.31)$$

with zero constant of integration. Substituting equations (2.5) into equation (3.31), we obtain

$$3(\epsilon^2+2)\left(ab(3-n)\chi^{2-n} + a^2(2-n)\chi^{3-2n} + nc^2\chi^{2n-1} + bc(n+1)\chi^n + (2ac+b^2)\chi\right) - 3\lambda\chi + \alpha\chi^3 = 0 \quad (3.32)$$

Setting n = 0, equation (3.32) is reduced to

$$3(\epsilon^2+2)\left(3ab\chi^2 + 2a^2\chi^3 + bc + (2ac+b^2)\chi\right) - 3\lambda\chi + \alpha\chi^3 = 0 \quad (3.33)$$

Putting each coefficient of χi(i = 0,1,2,3) to zero, we have

$$3(\epsilon^2+2)bc = 0,$$

$$(\epsilon^2+2)(2ac+b^2) - \lambda = 0$$

$$(\epsilon^2+2)ab = 0,$$

$$6(\epsilon^2+2)a^2 + \alpha = 0.$$

(3.34),(3.35),(3.36) and (3.37)

$$c = \lambda\sqrt{\frac{3}{-2(\epsilon^2+2)\alpha}}$$

$$a = \pm\sqrt{\frac{-\alpha}{6(\epsilon^2+2)}}.$$

Solving equations (3.34)-(3.37), we get b = 0,(3.38),(3.39) and(3.40)

Rational function solutions: (When b = 0 and c = 0)

Substituting equations (3.38)-(3.40) and (3.2) into equations (2.10) and (2.11), we obtain the exact wave soultions of equation (3.29),

$$\chi_1(x,t) = (-a(x+y+\epsilon z-\lambda t+\mu))^{-1} \quad (3.41)$$

Trigonometric function solution: (When λ > 0)

Substituting equations (3.38)-(3.40) and (3.30) into equations (2.10) and (2.11), we obtain the exact wave solutions of equation (3.29),

$$\chi_{2,3}(x,y,z,t) = \pm\sqrt{-\frac{3\lambda}{\alpha}}\, tan\left(\sqrt{\frac{\lambda}{2(\epsilon^2+2)}}(x+y+\epsilon z-\lambda t+\mu)\right)$$

$$\chi_{4,5}(x,y,z,t) = \pm\sqrt{-\frac{3\lambda}{\alpha}}\, cot\left(\sqrt{\frac{\lambda}{2(\epsilon^2+2)}}(x+y+\epsilon z-\lambda t+\mu)\right) \quad (3.42)\text{and } (3.43)$$

where and μ are arbitrary constants.

Hyperbolic function solution : (When λ < 0)

Substituting equations (3.38)-(3.40) and (3.2) into equations (2.12) and (2.13), we obtain the exact wave solutions of equation (3.29),

$$\chi_{6,7}(x,t) = \pm\sqrt{\frac{3\lambda}{\alpha}}\, tanh\left(\sqrt{-\frac{\lambda}{2(\epsilon^2+2)}}(x+y+\epsilon z-\lambda t+\mu)\right) \quad (3.44)\text{ and } (3.45)$$

$$\chi_{8,9}(x,t) = \pm\sqrt{\frac{3\lambda}{\alpha}}\, coth\left(\sqrt{-\frac{\lambda}{2(\epsilon^2+2)}}(x+y+\epsilon z-\lambda t+\mu)\right)$$

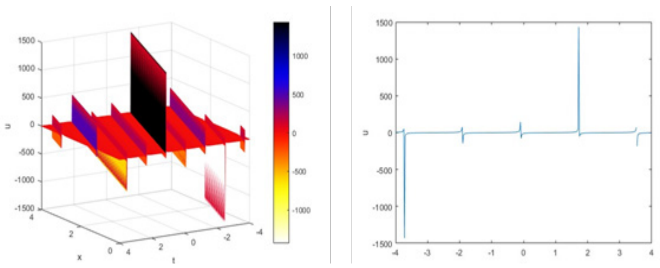
where and μ are arbitrary constants.

Remark 3.2. Applying equation (2.16) to χi(x,y,z,t) (i = 1,2,...,9) once, we can get an infinite sequence of solutions of equations (3.1) and (3.29).

GRAPHS FOR THE SOLUTIONS

In this section we give 3D graphics for some solutions, namely Figures. 1-6.

Figure 1: The soultion u=u_2(xyz) in (3.16) with w=-1.5, u=1, y=0, 0 ≤ t ≤ 4 and -4 ≤ x ≤4, the 3D plot on the left and the 2D plot



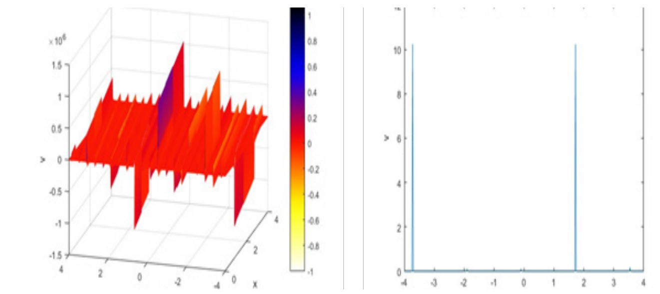


Figure 2: The solution v = v_2(x,y,t) in (3.17) with w = -1.5, μ =1, y=0, 0 ≤ t ≤ 4 and −4 ≤ x ≤ 4 , the 3D plot on the left and the 2D plot for t = 0 on the right.

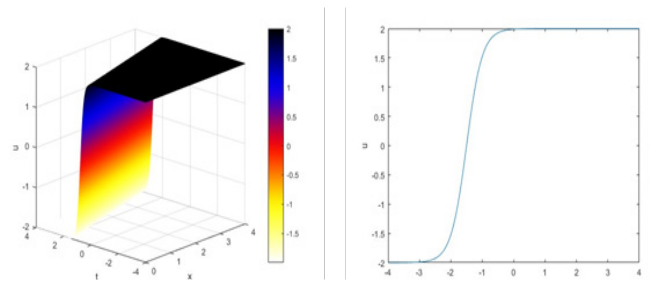


Figure 3: The solution u = u_6(x,y,t) in (3.20) with with w = 2, μ =1.5, y=0, 0 ≤ t ≤ 4 and −4 ≤ x ≤ 4 , the 3D plot on the left and the 2D plot for t = 0 on the right.

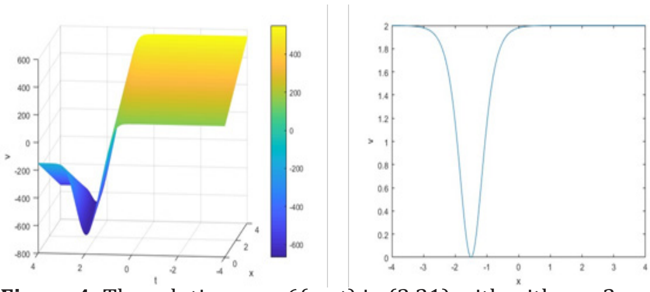


Figure 4: The solution u = v6(x,y,t) in (3.21) with with w = 2, μ =1.5, y=0, 0 ≤ t ≤ 4 and −4 ≤ x ≤ 4 , the 3D plot on the left and the 2D plot for t = 0 on the right.

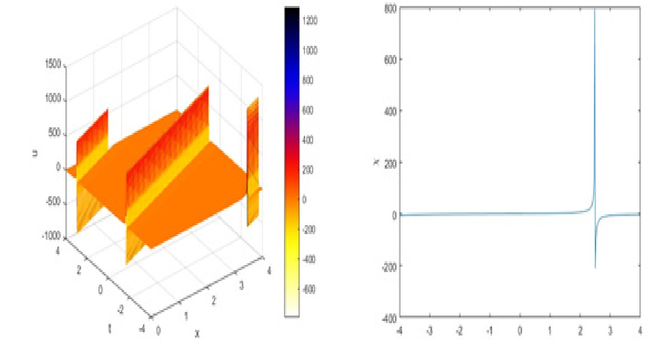


Figure 5: The solution χ = χ_2(x,y,z,t) in (3.42) with λ=1.5, ∈=1.3, μ=1, y=z=0, 0 ≤ t ≤ 4 and −4 ≤ x ≤ 4 , the 3D plot on the left and the 2D plot for t = 0 on the right.

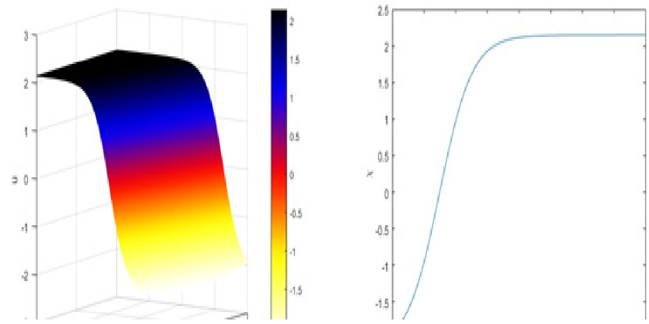


Figure 6: The solution χ = χ_6(x,y,z,t) in (3.44) with λ=-2, α = 1.3, ∈ =1.5, μ=1, y=z=0, 0 ≤ t ≤ 4 and −4 ≤ x ≤ 4 , the 3D plot on the left and the 2D plot for t = 0 on the right.

APPLICATIONS OF THE SOLUTIONS

The solutions of the Bogoyavlenskii equation (3.1) and the (3 + 1)-dimensional mKdV-ZK equation (3.29) are expressed by rational functions, trigonometric functions and hyperbolic functions. These solutions have great role in studying the Nonlinear Partial Differential Equations (NPDEs) and give information about character of differential equations [8]-[41]. Moreover the exact solutions may be worthwhile for explanation the mechanism of the complicated physical phenomena and dynamical processes modelled by these nonlinear partial differential equations. Apart from the physical relevance, the exact solutions of nonlinear partial differential equations help the numerical solvers to compare the correctness of their results and assist them in the qualitative analysis. Indeed the research on these nonlinear equations has increased in recent decades to gain a prudence through quantitative and qualitative characteristics of these equations. At the end, the exact solutions for the proposed equations in this article play an important role in many phenomena in physics such as solid state physics, plasma physics, fluid mechanics, hydrodynamics, optics and so on.

COMPARISONS

1. Here, we discuss the comparison between solutions given in [34, 35, 36] and our solutions. Najafi et al. [34] have introduced only six solutions for the Bogoyavlenskii equation, using the sine-cosine method. Whereas Alam and Tunc [35] given ten solutions of the Bogoyavlenskii equation, using the exp(−ϕ(ξ))-expansion method. Whereas Malik et al.

[36] employed the $(\frac{G'}{G})$-expansion method to this equation and obtained sixteen solutions. Moreover they introduced further sixteen solutions of the Bogoyavlenskii equation, using the generalized form of $(\frac{G'}{G})$-expansion method. Indeed, some of the solutions are similar to the results obtained by Peng et al. [37]. Actually these results much better the results given in [34,35,37]. One can observed that our results are newly constructed, although some of our have a similar structure to some of the existing results in the previous literature.

The main advantages of the Riccati-Bernoulli Sub-ODE method over the the sine-cosine method, the $(\frac{G'}{G})$-expansion method, the generalized form of $(\frac{G'}{G})$-expansion method and the exp$(-\phi(\xi))$-expansion method is that it supply many new exact traveling wave solutions along with additional free parameters. Indeed the Riccati-Bernoulli Sub-ODE method is more effective in providing many new solutions than the other methods.

2. On the other hand, we compare our result of the (3 + 1)-dimensional mKdV-ZK equation with the results given in [38, 39, 40]. Ilhan et al. [38] used the sine-Gordon expansion method and presented only five solutions. Islam et al. [39] introduced sixteen solutions, using the enhanced $(\frac{G'}{G})$-expansion method. Actually his proposed method is simple, flexible, easy to use and produces very accurate results. His result is much better than the result given in [38]. Naher [35] used the generalized and improved $(\frac{G'}{G})$-Expansion method, which provides more general and abundant of new exact traveling wave solutions with numerous free parameters. One can easily emphasis that the Riccati-Bernoulli SubODE method is robust, efficacious and adequate. Indeed, the obtained solutions in this work have some physical meaning which is related to the actual models, for example, the magnetic moment, the profile of a laminar jet and the rapidity of special relativity. Thus

we observed that these results turned out to be very useful in explaining the physical behavior of the models arising in the nonlinear science.

Based on this comparison, we deduce that the Riccati-Bernoulli Sub-ODE method superior to other methods.At the end, the Riccati-Bernoulli Sub-ODE technique has a very important feature, that admits infinite sequence of solutions of equation, which is explained clearly in Section 2.2. In fact this feature has never given for any other methods, given in [34-40].

CONCLUSIONS

In this work, the Riccati-Bernoulli Sub-ODE method is successfully applied to construct the new exact solutions of some nonlinear evolution equations of special interest in mathematical physics and nonlinear science. The validity of the methods has been tested by applying it successfully to the the Bogoyavlenskii equation and the (3 + 1)-dimensional mKdV-ZK equation. The proposed method can give a new infinite sequence of solutions. These solutions are expressed by rational functions, trigonometric functions and hyperbolic functions. We presented the 3D surfaces of all selected solutions, under the choice of suitable parameters. We carried out all the computations in this paper with the aid of the Matlab 17. Indeed, we show that the Riccati-Bernoulli Sub-ODE method superior to other methods.

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